Optimality and duality for multi-objective fractional programming in complex spaces

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Outline

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§1. Introduction

- Levinson[12] in 1966 was first studied for complex linear problem.
- There are many authors interested the dual problems for the kinds of the programming problems.
  (ex. linear / non-linear, non-fractional / fractional, etc.)
- Recently, we are interesting in complex multi-objective programming problems.
- First, we studied a general complex multi-objective programming:

$$\begin{array}{lll} \text{(CMP)} & \min & f(\zeta) = (f_1(\zeta), \cdots, f_p(\zeta)) \\ & \text{such that} & \zeta = (z, \bar{z}) \in X = \{\zeta \in Q \mid -g(\zeta) \in S\}, \end{array}$$

where $S \subset \mathbb{C}^q$ is a polyhedral cone, and $f : \mathbb{C}^{2n} \to \mathbb{C}^p$, $g : \mathbb{C}^{2n} \to \mathbb{C}^q$ are analytic in $\zeta = (z, \bar{z}) \in Q = \{(z, \bar{z}) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$.

[12] T.Y. Huang and T. Tanaka. Numerical Algebra, Control and Optimization. 2022, 12(1). 121-134.

What is our problem?

In this talk, we are going to consider the following complex multi-objective fractional programming problem,

$$\text{(CMFP)} \qquad \min \quad \left(\frac{\text{Re } f_1(\mathbf{z})}{\text{Re } g_1(\mathbf{z})}, \ldots, \frac{\text{Re } f_m(\mathbf{z})}{\text{Re } g_m(\mathbf{z})} \right)$$

$$\text{subject to} \quad \mathbf{z} \in X = \left\{ \mathbf{z} = (z, \bar{z}) \in \mathbb{C}^{2n} \mid -h(\mathbf{z}) \in S \right\},$$

where S is a polyhedral cone in $\mathbb{C}^p$; for $i = 1, \ldots, m$, $f_i(\cdot), g_i(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}$ and $h(\cdot) : \mathbb{C}^{2n} \to \mathbb{C}^p$ are analytic functions defined on $Q \subset \mathbb{C}^{2n}$, and $Q = \{\mathbf{z} = (z, \bar{z}) \mid z \in \mathbb{C}^n\} \subset \mathbb{C}^{2n}$ is a linear manifold over real field.

We want:

- Establish the optimality conditions theorems of problem (CMFP).
- Formulate the parametric dual problem and parametric free dual problems.

We give some notations:

1. Let $\mathbf{u} = (u_1, \ldots, u_n), \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$.
   - $\mathbf{u} = \mathbf{v}$ if and only if $u_i = v_i$, for all $i = 1, \ldots, n$.
   - $\mathbf{u} < \mathbf{v}$ if and only if $u_i < v_i$, for all $i = 1, \ldots, n$.
   - $\mathbf{u} \leq \mathbf{v}$ if and only if $u_i \leq v_i$, for all $i = 1, \ldots, n$.
   - $\mathbf{u} \leqq \mathbf{v}$ if and only if $\mathbf{u} \leq \mathbf{v}$ and $u_i \neq v_i$ for some $i \in \{1, \ldots, n\}$.

2. Given a general multi-objective programming (MP) as the following:

$$\text{(MP)} \quad \min \quad \phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \phi(\mathbf{x}), \ldots, \phi_m(\mathbf{x}))$$
$$s.t. \quad \mathbf{x} \in X,$$

where X be the feasible set and $\phi : X \to \mathbb{R}^m$ be the multi-function.

3. The optimality of the multi-objective minimization problem (MP) is given in the sense of Pareto optimality, which is defined as follows.

The point $\mathbf{x}_0 \in X$ *is said to be a* **minimal efficient solution** *(or Pareto minimal point) of problem (MP) if there does not exist other* $\mathbf{x} \in X$ *such that*

$$\phi(\mathbf{x}) \leqq \phi(\mathbf{x}_0).$$

4. Given $z \in \mathbb{C}^p$, the notations $\bar{z}$, z^T and z^H are the conjugate, transpose and conjugate transpose of z.
5. Let $S = \{z \in \mathbb{C}^p \mid \mathrm{Re}(Kz) \geq 0\} \subset \mathbb{C}^p$ be a polyhedral cone with matrix $K \in \mathbb{C}^{k \times p}$ where k is a positive integer.
6. The dual cone S^* of the convex cone S is defined by

$$S^* = \{\eta \in \mathbb{C}^p \mid \mathrm{Re}\langle z, \eta \rangle \geq 0 \text{ for } z \in S\},$$

where $\langle z, \eta \rangle = \eta^H z$ stands for the inner product in complex spaces.
7. For $s_0 \in S$, the set $S(s_0)$ is the intersection of those closed half spaces that includes s_0 in their boundaries. Thus if $s_0 \in \mathrm{int}(S)$, $S(s_0)$ is the whole space $\mathbb{C}^p$.

In order to establish the optimality conditions theorems and duality theorems, we need the following lemmas.

Lemma 1. (Differential representation)

For $\eta \in Y \subset \mathbb{C}^{2m}$, $w \in \mathbb{C}^n$ and $\zeta = (z, \bar{z}) \in Q \subset \mathbb{C}^{2n}$, we denote the function $\Phi(\zeta) = f(\zeta, \eta) + \langle h(\zeta), \mu \rangle$.

Then $\Phi(\zeta)$ is differentiable at $\zeta_0 = (z_0, \overline{z_0})$, and

$$Re[\Phi'(\zeta_0)(\zeta - \zeta_0)] = Re\left\langle z - z_0, \overline{\nabla_z f(\zeta_0, \eta)} + \nabla_{\bar{z}} f(\zeta_0, \eta) + \mu^T \overline{\nabla_z h(\zeta_0)} + \mu^H \nabla_{\bar{z}} h(\zeta_0) \right\rangle.$$

[8] H.C. Lai and T.Y. Huang, *Optimality conditions for a nondifferentiable minimax programming in complex spaces*, Nonlinear Analysis. 2009, 71, 1205-1212.

We could give the single-objective problems (CFP_i) and lemma as follows.
For $i = 1, \dots, m$, the single-objective problems (CFP_i) are defined by

$$(\mathrm{CFP}_i) \qquad \min \quad \frac{\mathrm{Re}\ f_i(\mathbf{z})}{\mathrm{Re}\ g_i(\mathbf{z})}$$

$$\text{s.t.} \quad \mathbf{z} \in M_i = \left\{ \mathbf{z} \in X \;\middle|\; \begin{array}{c} \dfrac{\mathrm{Re}\ f_j(\mathbf{z})}{\mathrm{Re}\ g_j(\mathbf{z})} \leq k_j, j = 1, \dots, m, \text{ with } j \neq i \\ \text{and } \; -h(\mathbf{z}) \in S \end{array} \right\}$$

$$= \left\{ \mathbf{z} \in X \;\middle|\; \begin{array}{c} \mathrm{Re}\ [f_j(\mathbf{z}) - k_j g_j(\mathbf{z})] \leq 0, j = 1, \dots, m, \text{ with } j \neq i \\ \text{and } \; -h(\mathbf{z}) \in S \end{array} \right\},$$

where $\mathbf{k} = (k_1, \dots, k_m) = \left(\dfrac{\mathrm{Re}\ f_1(\mathbf{z}_0)}{\mathrm{Re}\ g_1(\mathbf{z}_0)}, \dots, \dfrac{\mathrm{Re}\ f_m(\mathbf{z}_0)}{\mathrm{Re}\ g_m(\mathbf{z}_0)} \right) \in \mathbb{R}^m$, and S is a polyhedral cone in $\mathbb{C}^p$.

Lemma

$\mathbf{z}_0 \in \mathbb{C}^n$ be a minimal efficient solution of problem (CMFP)
$\Longleftrightarrow$ $\mathbf{z}_0$ solves (CFP$_i$), for all $i = 1, \ldots, m$.

Now, we drive the following necessary optimality conditions for problem (CMFP).

[11] T.Y. Huang and S.C. Ho (2021). Bull. Malays. Math. Sci. Soc.

Theorem 1. (Necessary Optimality Conditions)

Suppose that $\mathbf{z}_0$ is a minimal efficient solution of (CMFP) with optimal value $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{R}^m$, and let problem (CMFP) possesses the constraint qualification at $\mathbf{z}_0$. Then there are $\lambda_i \geq 0$ for $i = 1, \ldots, m$ with $\sum_{i=1}^m \lambda_i = 1$ and $\mu \in S^* \subset \mathbb{C}^p$ satisfied the following conditions.

$$\sum_{i=1}^{m} \lambda_i[(\overline{\nabla_z f_i(\mathbf{z}_0)} + \nabla_{\bar{z}} f_i(\mathbf{z}_0)) - k_i(\overline{\nabla_z g_i(\mathbf{z}_0)} + \nabla_{\bar{z}} g_i(\mathbf{z}_0))] + \mu^T \overline{\nabla_z h(\mathbf{z}_0)} + \mu^H \nabla_{\bar{z}} h(\mathbf{z}_0) = 0, \quad (1)$$

$$\text{Re } \mu^H h(\mathbf{z}_0) = 0. \quad (2)$$

In order to formulate the sufficient optimality conditions and duality theorems, we will introduce the generalized convexity in complex spaces as follows.

Definition

The real part of an analytic function $f(\cdot)$ from $\mathbb{C}^{2n}$ to $\mathbb{R}$ is called, respectively,

1. convex (strictly) at $\zeta_0 \in Q \subset \mathbb{C}^{2n}$ if
$$\text{Re}\,\left[f(\zeta) - f(\zeta_0)\right] \underset{(>)}{\geq} \text{Re}\,\left[f'(\zeta_0)(\zeta - \zeta_0)\right],$$
2. pseudoconvex (strictly) at $\zeta_0 \in Q$ if
$$\text{Re}\,\left[f'(\zeta_0)(\zeta - \zeta_0)\right] \geq 0 \Rightarrow \text{Re}\,\left[f(\zeta) - f(\zeta_0)\right] \underset{(>0)}{\geq 0},$$
3. quasiconvex at $\zeta_0 \in Q$ if
$$\text{Re}\,\left[f(\zeta) - f(\zeta_0)\right] \leq 0 \Rightarrow \text{Re}\,\left[f'(\zeta_0)(\zeta - \zeta_0)\right] \leq 0.$$

Theorem 2. (Sufficient Optimality Conditions)

Suppose that $\mathbf{z}_0$ is a feasible solution of (CMFP), and there exist $\lambda_i \geq 0$, $k_i \geq 0$, for $i = 1, \ldots, m$, $\mu \in S^* \subset \mathbb{C}^p$ satisfying conditions (1) and (2) in Necessary optimality Theorem. Assume that any one of the following conditions holds:

1. one of Re $\sum_{i=1}^{m} \lambda_i [f_i(\cdot) - k_i g_i(\cdot)]$ and Re$[\mu^H h(\cdot)]$ is strictly convex and another is convex at $\mathbf{z}_0 \in Q$, or both are strictly convex at $\mathbf{z}_0 \in Q$,
2. Re $\sum_{i=1}^{m} \lambda_i [f_i(\cdot) - k_i g_i(\cdot)]$ is quasiconvex at $\mathbf{z}_0 \in Q$ and Re $[\mu^H h(\cdot)]$ is strictly pseudoconvex at $\mathbf{z}_0 \in Q$,
3. Re $\sum_{i=1}^{m} \lambda_i [f_i(\cdot) - k_i g_i(\cdot)]$ is strictly pseudoconvex at $\mathbf{z}_0 \in Q$ and Re $[\mu^H h(\cdot)]$ is quasiconvex at $\mathbf{z}_0 \in Q$,
4. Re $\left\{ \sum_{i=1}^{m} \lambda_i [f_i(\cdot) - k_i g_i(\cdot)] + \mu^H h(\cdot) \right\}$ is strictly pseudoconvex at $\mathbf{z}_0 \in Q$.

Then $\mathbf{z}_0$ an efficient solution of (CMFP).

We will constitute the **parametric dual** problem (D) w.r.t. problem (CMFP) by using the Necessary optimality conditions theorem with some constraints.

$$\text{(D)} \qquad \max_{\mathscr{F}_D} \ \gamma = (r_1, \ \cdots, \ r_m),$$

where $\mathscr{F}_D$ is the set of all feasible solutions $(\lambda, \mathbf{u}, \mu, \gamma)$ subject to

$$\sum_{i=1}^{m} \lambda_i [(\overline{\nabla_z f_i(\mathbf{u})} + \nabla_{\bar{z}} f_i(\mathbf{u})) - r_i(\overline{\nabla_z g_i(\mathbf{u})} + \nabla_{\bar{z}} g_i(\mathbf{u}))] + \mu^T \overline{\nabla_z h(\mathbf{u})} + \mu^H \nabla_{\bar{z}} h(\mathbf{u}) = 0, \quad (3)$$

$$\mathrm{Re}[f_i(\mathbf{u}) - r_i g_i(\mathbf{u})] \geq 0, \ i = 1, \ldots, m, \quad (4)$$

$$\mathrm{Re}\langle h(\mathbf{u}), \mu \rangle \geq 0, \ \mu \neq 0 \text{ in } S^*, \quad (5)$$

for $\mathbf{u} = (u, \bar{u}) \in Q \subset \mathbb{C}^{2n}$, $\gamma = (r_1, \ldots, r_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$ in $\mathbb{R}^m$.

The duality theorems of (D) w.r.t. primary problem (CMFP) are established as the following. The first, we will prove that the feasible value of (CMFP) is not less than the feasible value of (D) under some suitable assumptions.

Theorem (Weak Duality)

Let $\mathbf{z} = (z, \bar{z})$ be (CMFP)-feasible solution, and $(\lambda, \mathbf{u}, \mu, \gamma)$ be (D)-feasible solution. If any one of the conditions (1), (2), (3), (4) in Sufficient optimality theorem holds, then there are **not exist** feasible solution $\mathbf{z} \in X$ of problem (CMFP) such that

$$\left(\frac{\text{Re } f_1(\mathbf{z})}{\text{Re } g_1(\mathbf{z})}, \ldots, \frac{\text{Re } f_m(\mathbf{z})}{\text{Re } g_m(\mathbf{z})} \right) \leqq \gamma.$$

Given an optimal efficient solution of problem (CMFP), we can obtain a feasible solution of the dual problem (D), and the following strong duality theorem will be proved.

Theorem (Strong Duality)

Suppose that $\mathbf{z}_0$ is a minimal efficient solution of (CMFP) with optimal value $\gamma = (r_1, \ldots, r_m)$. Then there exists $(\lambda, \mathbf{z}_0, \mu, \gamma)$ is a feasible solution of the dual problem (D). If the hypotheses of the weak duality theorem are fulfilled, then $(\lambda, \mathbf{z}_0, \mu, \gamma)$ is an optimal efficient solution of (D), and the two problems (CMFP) and (D) have the same optimal value.

If both optimal efficient solutions of primary problem (CMFP) and dual problem (D) are exist, then the optimal values of (CMFP) and (D) are equal under some assumptions. We could prove this result as the following theorem.

Theorem (Strictly Converse Duality)

Suppose that $\widehat{\mathbf{z}}$ is an optimal efficient solution of problem (CMFP) with optimal value $\widehat{\gamma} = (\widehat{r_1}, \ldots, \widehat{r_m})$, and $(\widehat{\lambda}, \widehat{\mathbf{u}}, \widehat{\mu}, \widehat{\gamma})$ is an optimal efficient solution of dual problem (D). Suppose that $\widehat{\mathbf{z}}$ is an optimal efficient solution of problem (CMFP) with optimal value $\widehat{\gamma} = (\widehat{r_1}, \ldots, \widehat{r_m})$, and $(\widehat{\lambda}, \widehat{\mathbf{u}}, \widehat{\mu}, \widehat{\gamma})$ is an optimal efficient solution of dual problem (D).

Assume that the assumptions of the strong theorem are fulfilled, and any one of the following conditions (1)-(4) holds:

1. *one of* $\operatorname{Re} \sum_{i=1}^{m} \widehat{\lambda}_i [f_i(\cdot) - k_i g_i(\cdot)]$ *and* $\operatorname{Re}[\widehat{\mu}^H h(\cdot)]$ *is strictly convex and another is convex at* $\widehat{\mathbf{u}} \in Q$, *or both are strictly convex at* $\widehat{\mathbf{u}} \in Q$,
2. $\operatorname{Re} \sum_{i=1}^{m} \widehat{\lambda}_i [f_i(\cdot) - k_i g_i(\cdot)]$ *is quasiconvex at* $\widehat{\mathbf{u}} \in Q$ *and* $\operatorname{Re} [\widehat{\mu}^H h(\cdot)]$ *is strictly pseudoconvex at* $\widehat{\mathbf{u}} \in Q$,
3. $\operatorname{Re} \sum_{i=1}^{m} \widehat{\lambda}_i [f_i(\cdot) - k_i g_i(\cdot)]$ *is strictly pseudoconvex at* $\widehat{\mathbf{u}} \in Q$ *and* $\operatorname{Re} [\widehat{\mu}^H h(\cdot)]$ *is quasiconvex at* $\widehat{\mathbf{u}} \in Q$,
4. $\operatorname{Re} \left\{ \sum_{i=1}^{m} \widehat{\lambda}_i [f_i(\cdot) - k_i g_i(\cdot)] + \widehat{\mu}^H h(\cdot) \right\}$ *is strictly pseudoconvex at* $\widehat{\mathbf{u}} \in Q$.

Then $\widehat{\mathbf{z}} = \widehat{\mathbf{u}}$, *and the optimal values of (CMFP) and (D) are equal.*

In order to construct the parametric free duality models for (CMFP), the necessary optimality theorem need some change as follows. Suppose that $\mathbf{z}_0$ is a minimal efficient solution of (CMFP) with optimal value $\mathbf{k} = (k_1, \ldots, k_m) \in \mathbb{R}^m$. Here $k_i = \dfrac{\mathrm{Re} f_i(\mathbf{z}_0)}{\mathrm{Re} g_i(\mathbf{z}_0)}$, $i = 1, \cdots, m$. From (1) of Theorem 1, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \lambda_i [(\overline{\nabla_z f_i(\mathbf{z}_0)} + \nabla_{\bar{z}} f_i(\mathbf{z}_0)) - k_i(\overline{\nabla_z g_i(\mathbf{z}_0)} + \nabla_{\bar{z}} g_i(\mathbf{z}_0))] \\ &\quad + \mu^T \overline{\nabla_z h(\mathbf{z}_0)} + \mu^H \nabla_{\bar{z}} h(\mathbf{z}_0) \\ &= \sum_{i=1}^{m} \lambda_i [(\overline{\nabla_z f_i(\mathbf{z}_0)} + \nabla_{\bar{z}} f_i(\mathbf{z}_0)) - \frac{\mathrm{Re} f_i(\mathbf{z}_0)}{\mathrm{Re} g_i(\mathbf{z}_0)} (\overline{\nabla_z g_i(\mathbf{z}_0)} + \nabla_{\bar{z}} g_i(\mathbf{z}_0))] \\ &\quad + \mu^T \overline{\nabla_z h(\mathbf{z}_0)} + \mu^H \nabla_{\bar{z}} h(\mathbf{z}_0). \end{aligned}$$

Replace λ_i by $\overline{\lambda_i}\ \mathrm{Re}[g_i(\mathbf{z}_0)]$, $i = 1, \ldots, m$ and from (2) of Theorem 1, we get

$$\sum_{i=1}^{m} \overline{\lambda_i}\ \left[\overline{\nabla_z f_i(\mathbf{z}_0)} + \nabla_{\bar{z}} f_i(\mathbf{z}_0) + \mu^T \overline{\nabla_z h(\mathbf{z}_0)} + \mu^H \nabla_{\bar{z}} h(\mathbf{z}_0)\right] \cdot \mathrm{Re}[g_i(\mathbf{z}_0)]$$
$$- \sum_{i=1}^{m} \overline{\lambda_i}\ \mathrm{Re}[f_i(\mathbf{z}_0) + \mu^H h(\mathbf{z}_0)] \cdot \left[\overline{\nabla_z g_i(\mathbf{z}_0)} + \nabla_{\bar{z}} g_i(\mathbf{z}_0)\right] = 0.$$

Then Theorem 1 change to

Theorem

Suppose that $\mathbf{z}_0$ is a minimal efficient solution of (CMFP), and let problem (CMFP) possesses the constraint qualification at $\mathbf{z}_0$. Then there are $\lambda_i \geq 0$ for $i = 1, \ldots, m$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $\mu \in S^* \subset \mathbb{C}^p$ satisfied the following conditions.

$$\sum_{i=1}^{m} \lambda_i\ \left[\overline{\nabla_z f_i(\mathbf{z}_0)} + \nabla_{\bar{z}} f_i(\mathbf{z}_0) + \mu^T \overline{\nabla_z h(\mathbf{z}_0)} + \mu^H \nabla_{\bar{z}} h(\mathbf{z}_0)\right] \cdot \mathrm{Re}[g_i(\mathbf{z}_0)]$$
$$- \sum_{i=1}^{m} \lambda_i\ \mathrm{Re}[f_i(\mathbf{z}_0) + \mu^H h(\mathbf{z}_0)] \cdot \left[\overline{\nabla_z g_i(\mathbf{z}_0)} + \nabla_{\bar{z}} g_i(\mathbf{z}_0)\right] = 0.$$

$$\mathrm{Re}\ \mu^H h(\mathbf{z}_0) = 0.$$

The Wolfe type dual model of (CMFP) is

$$(CMFD_W) \quad \max_{\mathscr{F}_{WD}} \left(\frac{\text{Re}\left[f_1(\mathbf{u}) + \mu^H h(\mathbf{u})\right]}{\text{Re}[g_1(\mathbf{u})]}, \cdots, \frac{\text{Re}\left[f_m(\mathbf{u}) + \mu^H h(\mathbf{u})\right]}{\text{Re}[g_m(\mathbf{u})]} \right)$$

where $\mathscr{F}_{WD}$ is the set of all feasible solutions $(\lambda, \mathbf{u}, \mu)$ subject to

$$\sum_{i=1}^{m} \lambda_i \left[\overline{\nabla_z f_i(\mathbf{u})} + \nabla_{\bar{z}} f_i(\mathbf{u}) + \mu^T \overline{\nabla_z h(\mathbf{u})} + \mu^H \nabla_{\bar{z}} h(\mathbf{u})\right] \cdot \text{Re}[g_i(\mathbf{u})]$$
$$- \sum_{i=1}^{m} \lambda_i \text{ Re}[f_i(\mathbf{u}) + \mu^H h(\mathbf{u})] \cdot \left[\overline{\nabla_z g_i(\mathbf{u})} + \nabla_{\bar{z}} g_i(\mathbf{u})\right] = 0.$$

$$\text{Re } \mu^H h(\mathbf{u}) = 0,$$

for $\mathbf{u} = (u, \bar{u}) \in Q \subset \mathbb{C}^{2n}$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $\mu \in S^* \subset \mathbb{C}^p$.

The Mond-Weir type dual model of (CMFP) is

$$(CMFD_{MW}) \qquad \max_{\mathscr{F}_{MWD}} \left(\frac{\mathrm{Re}\left[f_1(\mathbf{u})\right]}{\mathrm{Re}[g_1(\mathbf{u})]}, \cdots, \frac{\mathrm{Re}\left[f_m(\mathbf{u})\right]}{\mathrm{Re}[g_m(\mathbf{u})]} \right)$$

where $\mathscr{F}_{MWD}$ is the set of all feasible solutions $(\lambda, \mathbf{u}, \mu)$ subject to

$$\sum_{i=1}^{m} \lambda_i \left[\overline{\nabla_z f_i(\mathbf{u})} + \nabla_{\bar{z}} f_i(\mathbf{u})\right] \cdot \mathrm{Re}[g_i(\mathbf{u})] - \sum_{i=1}^{m} \lambda_i \; \mathrm{Re}[f_i(\mathbf{u})] \cdot \left[\overline{\nabla_z g_i(\mathbf{u})} + \nabla_{\bar{z}} g_i(\mathbf{u})\right] + \mu^T \overline{\nabla_z h(\mathbf{u})} + \mu^H \nabla_{\bar{z}} h(\mathbf{u}) = 0,$$

$$\mathrm{Re} \; \mu^H h(\mathbf{u}) = 0,$$

for $\mathbf{u} = (u, \bar{u}) \in Q \subset \mathbb{C}^{2n}$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $\mu \in S^* \subset \mathbb{C}^p$.

In order to construct **Mixed type duality model**, we take some notations as follows.

1. The constraint function in (CFP) is
   $h(\mathbf{z}) = \big(h_1(\mathbf{z}), h_2(\mathbf{z}), \cdots, h_p(\mathbf{z})\big) \in (-S) \subset \mathbb{C}^p$,
   the multiplier $\mu = (\mu_1, \ldots, \mu_p) \in S^* \subset \mathbb{C}^p$.
2. Partition the index set $P = \{1, \cdots, p\}$ of $h(\mathbf{z})$ to be
   $P = P_0 \cup P_1 \cup \cdots \cup P_t$ such that

$$Re \left\langle\, h_{P_r}(\mathbf{z}), \mu_{P_r} \,\right\rangle \leq 0 \text{ for } r = 0, 1, \ldots, t,$$

   where $h_{P_r}(\mathbf{z}) \equiv \big(h_i(\mathbf{z})\big)_{i \in P_r}$ and $\mu_{P_r} \equiv \big(\mu_i\big)_{i \in P_r}$.

3. $Re\ \langle h(\mathbf{z}),\mu\ \rangle = Re\ \langle\ h_{P_0}(\mathbf{z}),\mu_{P_0}\ \rangle + \sum_{r=1}^{t} Re\ \langle\ h_{P_r}(\mathbf{z}),\mu_{P_r}\ \rangle \leq 0.$

And for $r = 0,\ 1, \ldots,\ t,$

$$\langle h_{P_r}(\mathbf{z}),\mu_{P_r}\rangle = \sum_{i\in P_r}\mu_i h_i(\mathbf{z}),$$
$$Re\langle h'_{P_r}(\mathbf{z}_0)(\mathbf{z}-\mathbf{z}_0),\mu_{P_r}\rangle = Re\langle\ z - z_0\ ,\ \mu_{P_r}^T\overline{\nabla_z h_{P_r}(\mathbf{z}_0)} + \mu_{P_r}^H \nabla_{\bar{z}} h_{P_r}(\mathbf{z}_0)\ \rangle,$$

where $\mu_{P_r}^T$ stands for transpose of μ_{P_r} and $\mu_{P_r}^H = \overline{\mu_{P_r}^T}$ is the conjugate transpose of μ_{P_r}.

Mixed type dual: Considering the objective of fractional functional which added a part of the constraints of (CMFP) with a part of multiplier $\mu \in S^*$ (i.e. $Re \langle h_{P_0}(\mathbf{u}), \mu_{P_0} \rangle$) into the numerator of the fractional functional in (CFMP). Thus, the **Mixed type dual problem** of (CFMP) is

$$(CMFD_{Mix}) \quad \max_{\mathscr{F}_{Mix}} \left(\frac{\mathrm{Re}\left[f_1(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})\right]}{\mathrm{Re}[g_1(\mathbf{u})]}, \cdots, \frac{\mathrm{Re}\left[f_m(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})\right]}{\mathrm{Re}[g_m(\mathbf{u})]} \right)$$

where $\mathscr{F}_{Mix}$ is the set of all feasible solutions $(\lambda, \mathbf{u}, \mu)$ subject to

$$\sum_{i=1}^{m} \lambda_i \left[\overline{\nabla_z f_i(\mathbf{u})} + \nabla_{\bar{z}} f_i(\mathbf{u}) + \mu_{P_0}^T \overline{\nabla_z h_{P_0}(\mathbf{u})} + \mu_{P_0}^H \nabla_{\bar{z}} h_{P_0}(\mathbf{u})\right] \cdot \mathrm{Re}[g_i(\mathbf{u})]$$
$$- \sum_{i=1}^{m} \lambda_i \, \mathrm{Re}[f_i(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})] \cdot \left[\overline{\nabla_z g_i(\mathbf{u})} + \nabla_{\bar{z}} g_i(\mathbf{u})\right]$$
$$+ \sum_{r=1}^{t} [\mu_{P_r}^T \overline{\nabla_z h_{P_r}(\mathbf{u})} + \mu_{P_r}^H \nabla_{\bar{z}} h_{P_r}(\mathbf{u})] = 0.$$

$$\mathrm{Re}\ \mu_{P_r}^H h_{P_r}(\mathbf{u}) = 0, \quad r = 0, 1, \cdots, t,$$

for $\mathbf{u} = (u, \bar{u}) \in Q \subset \mathbb{C}^{2n}$ and $\lambda = (\lambda_1, \ldots, \lambda_m) \geq 0$ with $\sum_{i=1}^{m} \lambda_i = 1$ and $\mu \in S^* \subset \mathbb{C}^p$.

In problem (CMFD$_{Mix}$), if the index set P of the constraints in (CMFP) is separated by two parts P_0 and P_1, that is,

$$P = P_0 \cup P_1, \qquad (P_r = \emptyset \text{ for } r = 2, \cdots, t),$$

then the mixed type dual problem (CMFD$_{Mix}$) reduces to

(CMFD$_{Mix}$) $\equiv$ (CMFD$_W$), when $P_0 = P$ and $P_1 = \emptyset$ and
(CMFD$_{Mix}$) $\equiv$ (CMFD$_{MW}$), when $P_0 = \emptyset$ and $P_1 = P$.

This shows that the Wolfe type dual (CMFD$_W$) and the Mond-Weir type dual (CMFD$_{MW}$) are the special cases of the mixed type dual (CMFD$_{Mix}$).

Define

$$\Upsilon(\bullet) = \sum_{i=1}^{m} \lambda_i \, \mathrm{Re}[f_i(\bullet) + \mu_{P_0}^H h_{P_0}(\bullet)] \cdot \mathrm{Re}[g_i(\mathbf{u})] - \sum_{i=1}^{m} \lambda_i \, \mathrm{Re}[f_i(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})] \cdot \mathrm{Re}[g_i(\bullet)] + \sum_{r=1}^{t} \mathrm{Re}[\mu_{P_r}^H h_{P_r}(\bullet)]$$

Using some generalized convexities of $\Upsilon(\bullet)$, we will obtain the duality theorems.

Theorem (Weak Duality)

Let $\mathbf{z} = (z, \bar{z})$ *be (CMFP)-feasible solution, and* $(\lambda, \mathbf{u}, \mu)$ *be (CMFD*$_{Mix}$*)-feasible solution.*

If **some suitable conditions of** $\Upsilon(\mathbf{u})$ *hold,*
then there are **not exist** *feasible solution* $\mathbf{z} \in X$ *of problem (CMFP) such that*

$$\left(\frac{\mathrm{Re}\ f_1(\mathbf{z})}{\mathrm{Re}\ g_1(\mathbf{z})}, \ldots, \frac{\mathrm{Re}\ f_m(\mathbf{z})}{\mathrm{Re}\ g_m(\mathbf{z})} \right) \leqq \left(\frac{\mathrm{Re}\left[f_1(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})\right]}{\mathrm{Re}[g_1(\mathbf{u})]}, \cdots, \frac{\mathrm{Re}\left[f_m(\mathbf{u}) + \mu_{P_0}^H h_{P_0}(\mathbf{u})\right]}{\mathrm{Re}[g_m(\mathbf{u})]} \right).$$

Theorem (Strong Duality)

Suppose that $\mathbf{z}_0$ *is a minimal efficient solution of (CMFP).*
Then there exists $(\lambda, \mathbf{z}_0, \mu)$ *is a feasible solution of the dual problem* $(CMFD_{Mix})$.
If the hypotheses of above Theorem are fulfilled, then $(\lambda, \mathbf{z}_0, \mu)$ *is an optimal efficient solution of* $(CMFD_{MW})$, *and the two problems (CMFP) and* $(CMFD_{Mix})$ *have the same optimal value.*

Theorem (Strictly Converse Duality)

Suppose that $\widehat{\mathbf{z}}$ *is an optimal efficient solution of problem (CMFP), and* $(\widehat{\lambda}, \widehat{\mathbf{u}}, \widehat{\mu})$ *is an optimal efficient solution of dual problem (CMFD*$_{Mix}$*). Assume that the assumptions of above Theorem are fulfilled, and* **some suitable conditions of** $\Upsilon(\mathbf{u})$ *hold. Then* $\widehat{\mathbf{z}} = \widehat{\mathbf{u}}$*, and the optimal values of (CMFP) and (CMFD*$_{Mix}$*) are equal.*

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Thank you very much !!